JOURNAL OF COMPUTATIONAL PHYSICS 27, 71–87 (1978)

A Partition Technique for the Solution of Potential Flow Problems by Integral Equation Methods

AMIRAM HARTEN AND SHAMIR EFRONY

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

Received November 23, 1976; revised February 25, 1977

Discretization of integral equation methods for the computation of potential flows results in dense matrices. The partition technique of this paper is designed to remove this restriction on the size of problems that can be efficiently computed by integral equation methods. This technique consists of dividing the domain by partition curves and then applying a *standard integral equation method* to each of the subdomains. The potential on the partition curves as well as the absolute value of its normal derivative are considered to be additional unknowns. Although the number of unknowns was thus increased, the number of nonzero entries in the matrix of the extended system was considerably reduced. Using a partial elimination process in each of the subdomains, a determined system of equations for the unknowns on the partition curves is automatically derived. Once the latter is solved, the solution in each of the subdomains is obtained by back substitution.

The partition technique extends to problems in an unbounded domain with a radiation condition, and can be easily implemented in existing computer programs. To describe the technique, we choose the particular problem of time harmonic motion of floating bodies in two dimensions.

1. INTRODUCTION

In this paper we present a partition technique to improve the performance of integral equation methods in the computation of potential flows. The partition technique is general and can be applied to many problems of computational physics. Without loss of generality we describe the technique through a specific application, namely a two dimensional calculation of a linearized time harmonic motion of water due to a forced harmonic excitation of floating bodies.

The motion of the fluid is described by means of a velocity potential $\Phi(\mathbf{x}, t)$, $\mathbf{x} = (x, z)$ satisfying the Laplace equation

$$\Delta\Phi = \Phi_{xx} + \Phi_{zz} = 0 \tag{1.1}$$

and the linearized free surface condition on the undisturbed water surface Γ_F (see Fig. 1)

$$\Phi_{tt} + g\Phi_z = 0, \qquad \mathbf{x} \in \Gamma_F, \tag{1.2}$$

0021-9991/78/0271-0071$02.00/0

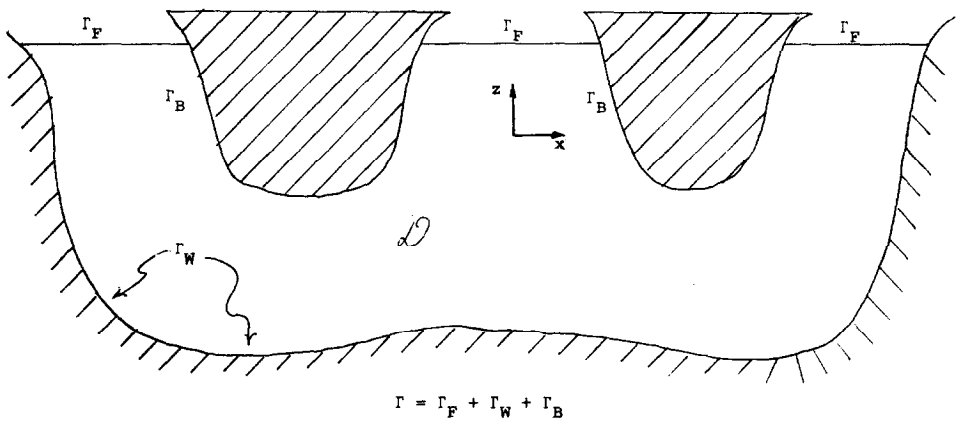


FIG. 1. Definition Sketch

where g is the gravitational acceleration. The free surface $z = \eta(x, t)$ is given by

$$\eta(x, t) = -\frac{1}{g}\Phi_t(\mathbf{x}, t), \qquad \mathbf{x} \in \Gamma_F . \tag{1.3}$$

The normal derivative of the potential on a rigid boundary coincides with the normal velocity of the boundary. On the fixed part of the boundary Γ_w

$$\Phi_n = 0, \qquad \mathbf{x} \in \Gamma_w . \tag{1.4a}$$

On the wetted part of the boundary of an oscillating floating body Γ_B

$$\begin{aligned} \Phi_n &= -n_x \frac{d}{dt} X - n_z \frac{d}{dt} Z + [(X - x_G) n_x - (Z - z_G) n_z] \frac{d}{dt} \Lambda \\ &\equiv -\omega[A''(x, z) \cos \omega t + A'(x, z) \sin \omega t] \\ &\equiv \text{Re } -i\omega A(x, z)\, e^{-i\omega t} \end{aligned} \tag{1.4b}$$

Here $\mathbf{n} = (n_x, n_z)$ is the unit normal pointing into the water domain $\mathcal{D}$; (x_G, z_G) is the center of gyration of the body and (X, Z) is its displacement from the rest position; Λ is the angle of the floating body measured from the vertical line; ω is the frequency of the oscillation.

The pressure $P(\mathbf{x}, t)$ is determined by Euler's integral

$$P + \rho\Phi_t - \rho g\hat{z} = 0 \tag{1.5}$$

with the term $\frac{1}{2}\rho \mid \nabla\Phi \mid^2$ suppressed as being of higher order; ρ is the density of the fluid; $\hat{z}$ is the distance of the point $\mathbf{x}$ from the water surface.

We suppose now that the body has been oscillating for such a long time that all transients have died out and that the velocity potential can be written as

$$\Phi(\mathbf{x}, t) = \mathrm{Re}\ -i\omega\phi(x, z)\, e^{-i\omega t}. \tag{1.6}$$

The function $\phi(\mathbf{x})$ satisfies Laplace's equation, the free surface condition

$$\phi_z - \lambda\phi = 0, \qquad \lambda = \frac{\omega^2}{g}, \qquad \mathbf{x} \in \Gamma_F, \tag{1.7}$$

and also

$$\frac{\partial\phi}{\partial n}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Gamma_W, \tag{1.8a}$$

$$\frac{\partial\phi}{\partial n}(\mathbf{x}) = A(x, z), \qquad \mathbf{x} \in \Gamma_B. \tag{1.8b}$$

Although the time-dependent potential Φ with bounded Φ, Φ_t and grad Φ is uniquely determined by its initial values, an additional condition is needed for the uniqueness of the steady state time-harmonic solution ϕ whenever the water domain $\mathscr{D}$ is unbounded. In case the fluid is of constant depth away from the body, this so called radiation condition states that asymptotically the solution is an outgoing wave (see [7, 8])

$$\lim(\phi_x \mp ik_0\phi) = 0 \qquad \text{as} \quad x \to \pm\infty \tag{1.9}$$

where the special wave number k_0 is the positive real root of the equation

$$k \tanh kH = \frac{\omega^2}{g} = \lambda. \tag{1.10}$$

In many applications only the solution on the boundary, namely the wave height (1.3), as well as the pressure exerted on the floating bodies and the confining walls, is of interest; thus only the potential on the boundary Γ of the water domain $\mathscr{D}$ need be obtained. An integral equation method based on Green's identity is tailored for this situation. This method easily handles curved boundaries as well as mixed boundary conditions, including the radiation condition at $\pm\infty$. The main computational effort in this method involves line integration (surface integration in $3D$). Therefore it is relatively easy to design high order accurate schemes. In Section 2 we describe integral equation methods based on Green's identity.

Seemingly, by reducing the dimensionality of the problem one gets a much smaller matrix approximation than that produced by finite elements methods on finite difference schemes. This statement is somewhat misleading: Using finite element methods one arrives at a matrix with a band structure. If N is the number of nodes along the perimeter, then the number of entries is proportional to N^4, but the number

of nonzero elements is proportional to N^2 only. On the other hand, integral equation methods produce a smaller matrix with const. N^2 entries but typically this matrix is dense. In Section 3 we present an artificial partition technique for integral equation methods which results in breaking the original problem into several subproblems, in a way which resembles introducing weak block structure to the original problem.

In Section 4 we extend the partition technique to problems in unbounded domains. This is done by efficiently matching a solution of an interior problem with that of an exterior problem.

In Section 5 we present numerical results which verify the validity of the partition technique and demonstrate the extra savings offered by it.

2. Integral Equation Methods for Potential Flow Problems

In this section we describe in general terms an integral equation method based on Green's identity for the solution of the potential on the boundary Γ of a two-dimensional domain $\mathscr{D}$. We refer the reader, interested in the history of applying these methods to water wave problems, to reviews in [8] and [10]. To simplify the description we consider first the interior problem in a bounded domain $\mathscr{D}$; discussion of the exterior problem is deferred to Section 4.

Let $\gamma(\mathbf{x}, \boldsymbol{\xi})$, $\boldsymbol{\xi} = (\xi, \zeta)$, be the following function.

$$\gamma(\mathbf{x}, \boldsymbol{\xi}) = \ln R + W(\mathbf{x}, \boldsymbol{\xi}). \tag{2.1}$$

Here $R^2 = (x - \xi)^2 + (z - \zeta)^2$ and $W(\mathbf{x}, \boldsymbol{\xi})$ is an arbitrarily chosen function which is harmonic in $\mathscr{D}$. Thus $\gamma(\mathbf{x}, \boldsymbol{\xi})$ satisfies the Laplace equation in $\mathscr{D}$ except at the point $\mathbf{x} = \boldsymbol{\xi}$ where it has a logarithmic singularity. Applying Green's identity to the velocity potential $\phi(\mathbf{x})$ we obtain

$$\alpha(\mathbf{x})\,\phi(\mathbf{x}) = \int_\Gamma \left[\gamma(\mathbf{x}, \boldsymbol{\xi}) \frac{\partial \phi}{\partial n}(\boldsymbol{\xi}) - \frac{\partial}{\partial n}\gamma(\mathbf{x}, \boldsymbol{\xi})\,\phi(\boldsymbol{\xi})\right] ds_\xi\,,$$

$$\mathbf{x} \in \mathscr{D} + \Gamma, \qquad \boldsymbol{\xi} \in \Gamma, \tag{2.2a}$$

where

$$\alpha(\mathbf{x}) = -\lim_{\epsilon \to 0} \int_{S_\epsilon} \frac{\partial \gamma}{\partial n}(\mathbf{x}, \boldsymbol{\xi})\, ds_\xi\,. \tag{2.2b}$$

Here S_ϵ is that part of a circle around $\mathbf{x}$ with an ϵ radius which is contained in $\mathscr{D}$ (see [11, pp. 252–258]).

The relation (2.2) when applied to a point $\mathbf{x}$ on the boundary Γ constitutes an integral equation for the boundary potential. The normal derivative of the potential on the boundary is either proportional to the potential (1.7) or is a known quantity

(1.8). With these boundary conditions the integral equation takes the following form:

$$\alpha(\mathbf{x})\,\phi(\mathbf{x}) + \int_\Gamma \left[\frac{\partial \gamma}{\partial n}(\mathbf{x}, \xi) + \Lambda(\xi)\,\gamma(\mathbf{x}, \xi)\right] \phi(\xi)\,ds_\xi = \int_{\Gamma_B} \gamma(\mathbf{x}, \xi)\,A(\xi)\,ds_\xi\,, \quad (2.3a)$$

$$\Lambda(\xi) = \begin{cases} \lambda, & \xi \in \Gamma_F\,, \\ 0, & \text{otherwise.} \end{cases} \quad (2.3b)$$

Equation (2.3) involves a singular kernel but its solution $\phi(\mathbf{x})$ is smooth except at corners. Therefore it seems worthwhile to use an accurate polynomial approximation to the potential and to integrate the resulting products of the polynomials times $\gamma(\mathbf{x}, \xi)$ analytically, at least in the neighborhood of the singularity. We chose to discretize (2.3) by using a cubic spline approximation to $\phi(\mathbf{x})$.

Let $\mathbf{X}_j$, $j = 1,..., N$ be a partition of the boundary Γ and let $\phi_j = \phi(\mathbf{X}_j)$. The points $\mathbf{x}_j$ are not uniformly distributed; $\Delta_j = |\,\mathbf{x}_{j+1} - \mathbf{x}_j\,|$ should be smaller towards corners. The variation of ϕ in the z-direction is typically that of $\cosh kz$, while in the x-direction it behaves like $\cos kx$. Consequently one should use a partition which depends on the wave number k; typically one uses more partition points per unit length in the x-direction than in the z-direction.

Let $S_j(\mathbf{x}(s))$ be a one-dimensional unit cubic spline

$$S_j(\mathbf{x}_i) = \delta_{ij} \qquad \text{(the Kronecker-}\delta\text{)}. \quad (2.4)$$

In case the boundary has corners, we construct $S_j(\mathbf{x})$ so that it will be continuous up to the second derivative in a smooth segment containing $\mathbf{x}_j$, and to be identically zero in all other segments. This way, the approximation

$$f(\mathbf{x}) \sim \sum_{j=1}^{N} S_j(\mathbf{x})\,f(\mathbf{x}_j) \quad (2.5)$$

is continuous up to the second derivative, except at corners where it is only continuous.

Approximating $\phi(\mathbf{x})$ in the integral equation (2.3) by (2.5) results in the following matrix equations

$$G\boldsymbol{\phi} = \mathbf{a} \quad (2.6a)$$

where

$$\boldsymbol{\phi} = (\phi_1,..., \phi_N)^T, \qquad \mathbf{a} = (a_1,..., a_N)^T, \quad (2.6b)$$

$$G_{ij} = \alpha(\mathbf{x}_i)\,\delta_{ij} + \int_\Gamma \left[\frac{\partial \gamma}{\partial n}(\mathbf{x}_i, \xi) + \Lambda(\xi)\,\gamma(\mathbf{x}_i, \xi)\right] S_j(\xi)\,ds_\xi\,, \quad (2.6c)$$

$$a_i = \int_{\Gamma_B} \gamma(\mathbf{x}_i, \xi)\,A(\xi)\,ds_\xi\,. \quad (2.6d)$$

Equation (2.3) holds for all $\gamma(\mathbf{x}, \xi)$ of the form (2.1) and we are free to choose any harmonic function $W(\mathbf{x}, \xi)$. If one chooses $W(\mathbf{x}, \xi)$ so that for $\xi \in \Gamma_0 \subset \Gamma$

$$\frac{\partial \gamma}{\partial n}(\mathbf{x}, \xi) + \Lambda(\xi)\,\gamma(\mathbf{x}, \xi) = 0 \qquad \text{for all} \quad \mathbf{x} \in \Gamma \quad (2.7)$$

then the integral along Γ in the left hand side of equation (2.3) can be written as an integral along $\Gamma - \Gamma_0$ only and a smaller system for the discrete values of ϕ along $\Gamma - \Gamma_0$ obtains. At first glance this approach seems to be computationally efficient, however numerical tests indicate otherwise [4]. The reason for that is the complexity of the Green function associated with (2.7) and the fact that in this particular method most of the computing time goes into calculating the coefficients (2.6c). Therefore we shall take $W(\mathbf{x}, \xi) = 0$ (the same choice was made in [9]) except when the bottom is flat; then we take $W(\mathbf{x}, \xi) = \log \hat{R}$, where $\hat{R}$ is the distance between $\mathbf{x}$ and ξ; ξ is the reflection of ξ with respect to the flat bottom. This choice of a simple $\gamma(\mathbf{x}, \xi)$ enables us to compute analytically the coefficients (2.6).

We observe that with this choice of $\gamma(\mathbf{x}, \xi)$ the integrals

$$\int_\Gamma \frac{\partial \gamma}{\partial n}(\mathbf{x}, \xi)\, S_j(\xi)\, ds_\xi \qquad \text{and} \qquad \int_{\Gamma_F} \gamma(\mathbf{x}, \xi)\, S_j(\xi)\, ds_\xi$$

depend only on the geometry of the problem. This fact can be used to compute several modes of oscillation with several frequencies simultaneously [4].

The same approach can be carried over to time-dependent problems [2, 3].

3. An Artificial Partition Technique

Matrices obtained from a discretization of the integral equation (2.3) are dense. Consequently in a computation of the boundary potential due to a high frequency oscillation of the floating bodies, one can very easily run into problems of insufficient computer memory as well as of a large computing time. It is possible to overcome this problem by introducing a sort of block structure to the original matrix. This is accomplished by an artificial division of the given domain $\mathscr{D}$ into several subdomains $\mathscr{D}_1, \ldots, \mathscr{D}_K$ (see Fig. 2).

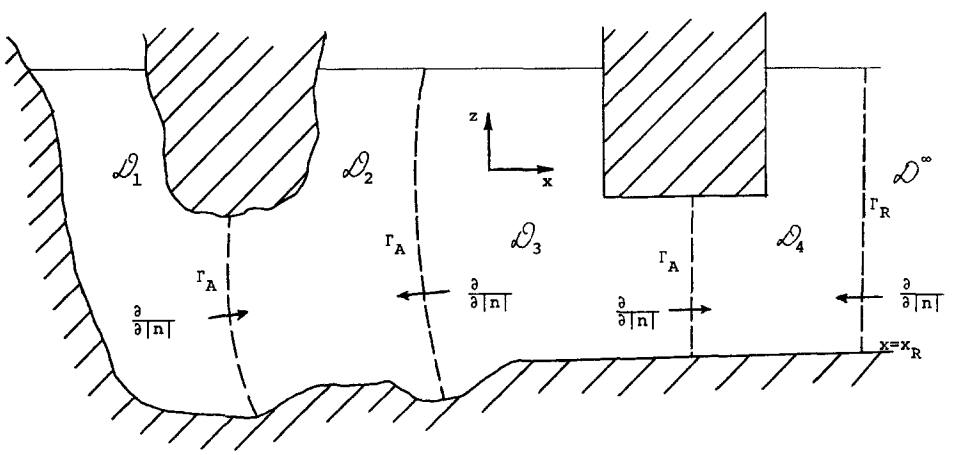


FIG. 2. Partition and Matching

In the following we apply relation (2.2) to each subdomain $\mathscr{D}_k$, where both ϕ and $\partial\phi/\partial n$ are unknown. The solution of the original problem in $\mathscr{D}$ is obtained by requiring continuity of the potential and its normal derivative across the partition curves Γ_A (see [6]). Since both sides of the partition curves take part in the formulation, one has to specify a direction of the normal on Γ_A; differentiation with respect to this prescribed direction is denoted by $\partial/\partial\,|\,n\,|$.

Let $\mathbf{x}_j$, $j = 1,\ldots, N$, be the division points on the boundary Γ of $\mathscr{D}$, as before, and let $\mathbf{x}_j$, $j = N+1,\ldots, N+N_A$ be some ordering of the points on the partition curves Γ_A. We introduce a new vector ψ of $N + 2N_A$ unknowns

$$\psi_l = \begin{cases} \phi(\mathbf{x}_l), & 1 \leqslant l \leqslant N + N_A, \\ \dfrac{\partial}{\partial\,|\,n\,|}\,\phi(x_{l-N_A}), & N + N_A + 1 \leqslant l \leqslant N + 2N_A. \end{cases} \tag{3.1}$$

A similar ordering is used in the subdomains $\mathscr{D}_k$; in the following we shall use superscripts of k to indicate intersection with $\mathscr{D}_k$. We denote by $\mathbf{x}_j^{(k)}$, $j = 1,\ldots, N^{(k)} + N_A^{(k)}$, those $\mathbf{x}_j$ which are on $\Gamma^{(k)}$, the boundary of $\mathscr{D}_k$. This sequence is ordered in such a way that $\mathbf{x}_j^{(k)}$, $j = 1,\ldots, N^{(k)}$, correspond to points on Γ, while $x_j^{(k)}$, $j = N^{(k)} + 1,\ldots, N^{(k)} + N_A^{(k)}$, correspond to points on $\Gamma_A^{(k)}$, the partition curves in the boundary of $\mathscr{D}_k$. We denote by $\psi^{(k)}$ the $N^{(k)} + 2N_A^{(k)}$ vector of unknowns

$$\psi_l^{(k)} = \begin{cases} \phi(x_l^{(k)}), & 1 \leqslant l \leqslant N^{(k)} + N_A^{(k)}, \\ \dfrac{\partial}{\partial\,|\,n\,|}\,\phi(x_{l-N_A^{(k)}}^{(k)}), & N^{(k)} + N_A^{(k)} + 1 \leqslant l \leqslant N^{(k)} + 2N_A^{(k)}. \end{cases} \tag{3.2}$$

Now we can specify $\partial/\partial\,|\,n\,|$ on Γ_A by prescribing that the normal to $\Gamma_A^{(2k)} = \Gamma_A \cap \mathscr{D}_{2k}$ is pointing into $\mathscr{D}_{2k}$ (see Fig. 2). Next we apply relation (2.2) to the subdomain $\mathscr{D}_k$; there, according to the above sign convention,

$$\frac{\partial}{\partial n} = (-1)^k \frac{\partial}{\partial\,|\,n\,|}\,.$$

$$\begin{aligned} \alpha(\mathbf{x})\,\phi(\mathbf{x}) + \int_{\Gamma^{(k)}} \left[\frac{\partial}{\partial n}\,\gamma(\mathbf{x}, \boldsymbol{\xi}) + \Lambda(\boldsymbol{\xi})\,\gamma(\mathbf{x}, \boldsymbol{\xi})\right] \phi(\boldsymbol{\xi})\, ds_\xi \\ + (-1)^k \int_{\Gamma_A^{(k)}} \gamma(\mathbf{x}, \boldsymbol{\xi})\, \frac{\partial}{\partial\,|\,n\,|}\,\phi(\boldsymbol{\xi})\, ds_\xi \\ = \int_{\Gamma_B^{(k)}} \gamma(\mathbf{x}, \boldsymbol{\xi})\, A(\boldsymbol{\xi})\, ds_\xi\,. \end{aligned} \tag{3.3}$$

Writing this relation for all $x_j^{(k)}$, $j = 1,\ldots, N^{(k)} + N_A^{(k)}$, we obtain the following underdetermined system of $N^{(k)} + N_A^{(k)}$ linear equations with $N^{(k)} + 2N_A^{(k)}$ unknowns:

$$G^{(k)}\psi^{(k)} = \mathbf{a}^{(k)} \tag{3.4a}$$

$$G_{l,j}^{(k)} = \alpha(\mathbf{x}_l)\,\delta_{l,j} + \int_{\Gamma^{(k)}} \left[\frac{\partial \gamma}{\partial n}\,(\mathbf{x}_l\,, \boldsymbol{\xi}) + \Lambda(\boldsymbol{\xi})\,\gamma(\mathbf{x}_l\,, \boldsymbol{\xi})\right] S_j(\boldsymbol{\xi})\, ds_\xi$$

$$\text{for} \quad 1 \leqslant l \leqslant N^{(k)} + N_A^{(k)}, \quad 1 \leqslant j \leqslant N^{(k)} + N_A^{(k)}, \tag{3.4b}$$

$$G_{l,j}^{(k)} = (-1)^k \int_{\Gamma_A^{(k)}} \gamma(\mathbf{x}_l, \xi)\, S_{j-N_A}(\xi)\, ds_\xi$$

$$\text{for} \quad 1 \leqslant l \leqslant N^{(k)} + N_A^{(k)}, \quad N^{(k)} + N_A^{(k)} + 1 \leqslant j \leqslant N^{(k)} + 2N_A^{(k)}, \tag{3.4c}$$

$$a_l^{(k)} = \int_{\Gamma_B} \gamma(\mathbf{x}_l, \xi)\, A(\xi)\, ds_\xi . \tag{3.4d}$$

This is very similar to the algorithm given in Section 2 and the same computer program can be used.

We proceed by eliminating $\psi_j^{(k)}$, $1 \leqslant j \leqslant N^{(k)}$, the discrete values of the potential along $\Gamma^{(k)} - \Gamma_A^{(k)}$ from the last $2N_A^{(k)}$ equations, using a Gaussian elimination procedure with row pivoting. The resulting system of linear equation is of the form

$$\begin{matrix} \uparrow \\ N^{(k)} \\ \downarrow \\ \uparrow \\ N_A^{(k)} \\ \downarrow \end{matrix} \begin{bmatrix} U^{(k)} & C^{(k)} & D^{(k)} \\ 0 & & \\ \hline 0 & A^{(k)} & B^{(k)} \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}^{(k)} \\ \hline \boldsymbol{\phi}_A^{(k)} \\ \hline \dfrac{\partial}{\partial |n|} \boldsymbol{\phi}_A^{(k)} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{(k)} \\ \hline \mathbf{a}_A^{(k)} \end{bmatrix} . \tag{3.5}$$

$$\leftarrow N^{(k)} \rightarrow\!\leftarrow N_A^{(k)} \rightarrow\!\leftarrow N_A^{(k)} \rightarrow$$

The subscript A denotes values belonging to the artificial partition curves. The last $2N_A^{(k)}$ equations involve only values on the partition curves. At this point we can store the first $N^{(k)}$ equations in an external device. The last $2N_A^{(k)}$ equations of each subdomain $\mathscr{D}_k$ are stored internally in proper locations of a matrix G_A of the dimension $2N_A \times 2N_A$ such that

$$G_A \begin{bmatrix} \psi_{N+1} \\ \vdots \\ \psi_{N+2N_A} \end{bmatrix} = \mathbf{a}_A . \tag{3.6}$$

Every point on the partition curves introduces two additional unknowns and contributes a single equation to (3.4) for each subdomain $\mathscr{D}_k$ to which it belongs. In the context of this paper we use partition points which belong exactly to two neighboring subdomains. Therefore the process of accumulating the last $2N_A^{(k)}$ equations in (3.5) results in the determined system (3.6). The solution of (3.16), the values of the potential and its normal derivative on the partition curves, is now used to solve the upper triangular systems

$$U^{(k)}\boldsymbol{\phi}^{(k)} = \mathbf{a}^{(k)} - C^{(k)}\boldsymbol{\phi}_A^{(k)} - D^{(k)} \frac{\partial}{\partial |n|} \boldsymbol{\phi}_A^{(k)} . \tag{3.7}$$

These systems are solved by inexpensive back substitution.

For the particular problem discussed in this paper, it is advisable to use partition curves which are almost vertical since only few points are needed to describe accurately the variation of the potential in the z-direction.

Although the introduction of the partition curves increased the number of unknowns (presumably by a small number) and consequently the size of the matrix corresponding to the enlarged system, it decreased considerably the number of the nonzero entries. The matrix of the extended system has nonzero diagonal blocks corresponding to $\phi^{(k)}$, the boundary potential values in $\mathscr{D}_k$, which are coupled by small off diagonal blocks corresponding to values on the partition curves. For large frequencies the number of points along vertical partition lines is small compared with the total number of points on the boundary of the subdomains. Suppose that $\mathscr{D}$ is divided into three subdomains containing each $N/3$ points. Then the number of nonzero entries is roughly $N^2/3$. The number of operations needed to compute these entries and to execute the elimination process is roughly 1/9 as large as in the original problem.

In Section 5 we shall demonstrate the efficiency of the partition technique by numerical examples.

4. Problems in an Unbounded Domain

In this section we shall discuss approximate solution to floating body problems in which the domain is unbounded in the x-direction. To obtain uniqueness in this case one has to impose a radiation condition of the form (1.9) which states that asymptotically the solution is an outgoing wave.

John [5] gives an explicit formula for a Green function $\gamma^R(\mathbf{x}, \xi)$ which satisfies the radiation condition (1.9) as well as

$$\gamma_z{}^R - \lambda\gamma^R = 0 \qquad \text{for} \quad z = H, \tag{4.1a}$$

$$\gamma_z{}^R = 0 \qquad \text{for} \quad z = 0. \tag{4.1b}$$

Setting $R = |x - \xi|$, this Green function is given by

$$\gamma^R((x, z), (\xi, \zeta)) \tag{4.2}$$

$$= \frac{1}{2}\log[R^2 + (z - \zeta)^2] + \frac{1}{2}\log[R^2 + (z + \zeta)^2] - 2\log H$$

$$-2\int_0^\infty \left[\frac{(\lambda + \mu)\exp(-\mu H)\cosh \mu H \cosh \mu z \cos \mu R}{\mu(\mu \sinh \mu H - \lambda \cosh \mu H)} + \frac{\exp(-\mu H)}{\mu}\right] d\mu.$$

The path of integration in the complex μ-plane is shown in Fig. 3. This Green function (4.2) has the following series representation (see [5])

$$\gamma^R = -2\pi i \sum_{m=0}^{\infty} \frac{K_m}{k_m} \cosh k_m z \cosh k_m \zeta \exp(-|k_m| R) \tag{4.3a}$$

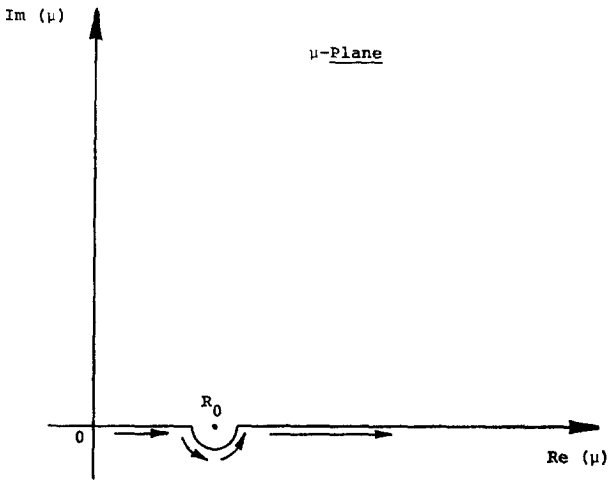


FIG. 3. Path of integration in (4.2).

where k_0 is the positive real root and k_m, $m \geqslant 1$ are the positive pure imaginary roots of the Eq. (1.10) and

$$K_m = \frac{k_m^2 - \lambda^2}{H(k_m^2 - \lambda^2) + \lambda}, \qquad m \geqslant 0. \tag{4.3b}$$

Using the Green function (4.2) or (4.3) in the integral equation (2.3) involves only the boundaries of the floating bodies and the part of the sea bottom which is different from the constant depth H. However, this function is rather complicated and its computation is time consuming. The matrix of the discrete approximation has complex entries, and therefore it requires a double storage and slower arithmetic operations. It is our opinion that this approach will be superior to other methods only if the geometry is relatively simple.

The radiation condition (1.9) can be approximated by imposing

$$\frac{\partial \phi}{\partial x} = \pm ik_0 \phi \qquad \text{at} \qquad x = \pm |x_R|, \tag{4.4}$$

where k_0 is the wave number given by (1.10) and x_R is a cut off location sufficiently far from the floating bodies. The approximation (4.4) was tested by Bai [1] in a variational method; x_R was taken to be so that the cut off location will be at least $5H$ away from the floating bodies.

The same approximate boundary condition was used by Ho and Harten in [4] in an integral equation formulation. There, the Green's relation (2.2) is applied to a domain bounded by an artificial vertical boundary Γ_R located at $x = x_R$ on which condition (4.4) is imposed, and the same simple $\gamma = \log R$ is used. This allows a fast computation of the matrix coefficients. On the other hand a portion of the entries is

complex and many unnecessary points have to be added in order to approximate the radiation condition.

Large scale calculations in which there is both a complicated geometry and a radiation condition can be made economically possible only if one can use a simple and real $\gamma(\mathbf{x}, \xi)$ in the interior and match it efficiently with the solution of the exterior problem. Yeung [10] suggests matching the solution to the integral equation of the interior with an eigenfunction expansion of the solution to the exterior problem. In the following we shall describe a straightforward extension of the partition idea of Section 3 to problems in unbounded domains. This technique is generalized by admitting a partition of the domain which contains unbounded subdomains; in the latter γ is taken to be the Green function γ^R defined by the eigenfunction expansion (4.3). The matching procedure provided by this approach is basically the same as the one suggested in [10] except that it is higher order accurate. However, the matching is carried out as a part of the partition technique, using the same efficient algorithm.

The interior is separated from the exterior by vertical boundaries, Γ_R, located at some convenient distance away from the floating bodies $x = x_R$; *this distance need not be large.* We proceed by using the technique described in Sections 2 and 3 in the domain bounded by Γ_R as if Γ_R was a partition curve; i.e., both ϕ and $\partial\phi/\partial n$ on Γ_R are considered as unknowns. The $2N_R$ values of ϕ and $\partial\phi/\partial n$ corresponding to the matching boundaries Γ_R are ordered at the end of the vector of unknowns ψ (3.1). Thus we define the new vector of unknowns $\tilde{\psi}$

$$\tilde{\psi}_l = \begin{cases} \psi_l, & 1 \leqslant l \leqslant \tilde{N}, \\ \phi(\mathbf{x}_l), & \tilde{N}+1 \leqslant l \leqslant \tilde{N}+N_R, \\ \dfrac{\partial}{\partial n}\phi(\mathbf{x}_{l-\tilde{N}}), & \tilde{N}+N_R+1 \leqslant l \leqslant \tilde{N}+2N_R, \end{cases} \tag{4.5}$$

where

$$\tilde{N} = N + 2N_A .$$

Using the algorithm (3.3), (3.4) and (3.5) we arrive at the *underdetermined* system of linear equations

$$\hat{G}_A \begin{bmatrix} \psi_{N+1} \\ \vdots \\ \psi_{\tilde{N}} \\ \tilde{\psi}_{\tilde{N}+1} \\ \vdots \\ \tilde{\psi}_{\tilde{N}+2N_R} \end{bmatrix} = \mathbf{a}_A , \tag{4.6}$$

where $\mathbf{a}_A$ is of the dimension $2N_A + N_R$ and the matrix $\tilde{G}_A$ is of the dimension $(2N_A + N_R) \times (2N_A + 2N_R)$. We proceed by eliminating $\psi_{N+1}, \ldots, \psi_{N+2N_A}$ from the last N_R equations using a Gaussian elimination procedure with row pivoting. As in (3.5), the resulting system of linear equations is of the form

$$\begin{array}{c} \uparrow \\ 2N_A \\ \downarrow \\ \uparrow \\ N_R \\ \downarrow \end{array} \left[\begin{array}{c|c|c} \begin{array}{cc} & U_A \\ 0 & \end{array} & C_A & D_A \\ \hline 0 & A_R & B_R \end{array}\right] \left[\begin{array}{c} \phi_A \\ \hline \phi_R \\ \hline \dfrac{\partial}{\partial n}\phi_R \end{array}\right] = \left[\begin{array}{c} \tilde{a}_A \\ \hline \tilde{a}_R \end{array}\right] \qquad (4.7)$$

$$\longleftarrow 2N_A \longrightarrow \longleftarrow N_R \longrightarrow \longleftarrow N_R \longrightarrow$$

where U_A is an upper triangular matrix and the subscript R is used to denote values belonging to the matching boundaries Γ_R. At this stage the first $2N_A$ equations can be stored on an external device. Up to this point all matrices are real and $\gamma(\mathbf{x}, \boldsymbol{\xi})$ is a simple and real function.

The next step is to supplement the last N_R equations with the $2N_R$ unknowns in (4.7)

$$A_R\phi_R + B_R \frac{\partial}{\partial n}\phi_R = \mathbf{a}_R \qquad (4.8)$$

by the additional N_R equations obtained from the formulation of the exterior problem(s) in the strip(s) $0 \leqslant z \leqslant H$, $x_R < x < +\infty$ $(-\infty < x < -x_R)$.

$$\pi\phi^R(x_R, z_j) = -\int_0^H \gamma^R((x_R, z_j), (x_R, \zeta)) \frac{\partial\phi_R}{\partial n}(x_R, \zeta)\, d\zeta \qquad (4.9)$$

where γ^R is the complex Green function (4.2), and the normal to Γ_R is pointing into the interior domain. Observe that because Γ_R is vertical there are no contributions from the term $\phi^R(\partial\gamma^R/\partial n)$ other than the left-hand side of Eq. (4.9). The determined system of Eqs. (4.8), (4.9) of $2N_R$ equations with $2N_R$ unknowns is now solved for the complex potential ϕ_R. Next the $(2N_A) \times (2N_A)$ upper triangular system of Eqs. (4.7) is read in and solved by back substitution for ϕ_A; from here we proceed to solve the systems (3.5) as described in Section 3.

We remark that equation (4.9) is approximated by using the series representation (4.3) and a spline approximation for $\partial\phi/\partial n$. The high order of smoothness of the cubic splines enables us to truncate the series after few terms (usually 10 or less). This approximation is described in an appendix.

5. Numerical Results and Conclusions

The partition and matching techniques presented in this paper speed up computation and decrease the in core memory requirements; more than that, these techniques are essential for solving large scale problems by integral equation methods.

The programming logic of these techniques is rather simple and can be easily

incorporated in an existing computer code for the solution of the potential on the boundary of a single closed curve. The numerical results reported in this section were obtained from a modification of the computer code TWODIM described in [4]. TWODIM contains a subroutine which computes the coefficients of the matrix equation (2.6) of a given curve Γ. All we had to do in order to program the partition technique of Section 3 was to add a subroutine which defines the closed curve $\Gamma^{(k)}$, and its correspondence to the original curve Γ. Then the subroutines of TWODIM were applied to the new closed curve $\Gamma^{(k)}$ and the coefficients (3.4) were stored in the proper locations.

We have compared numerical results of the partition and matching techniques with those obtained by TWODIM; the latter is a well verified program (see [4]). The test cases were those of rectangular platforms of various sizes floating in a bounded or semi-bounded rectangular channel or in an open sea. We have found a fairly close agreement in the numerical results of both programs. The results shown in Figs. 4 and 5 are a typical sample of this agreement. In these figures we plot μ and δ

$$\mu = -\mathrm{Re} \int_{\Gamma_P} \phi V_n \, ds, \tag{5.1a}$$

$$\delta = -\mathrm{Im} \int_{\Gamma_P} \phi V_n \, ds, \tag{5.1b}$$

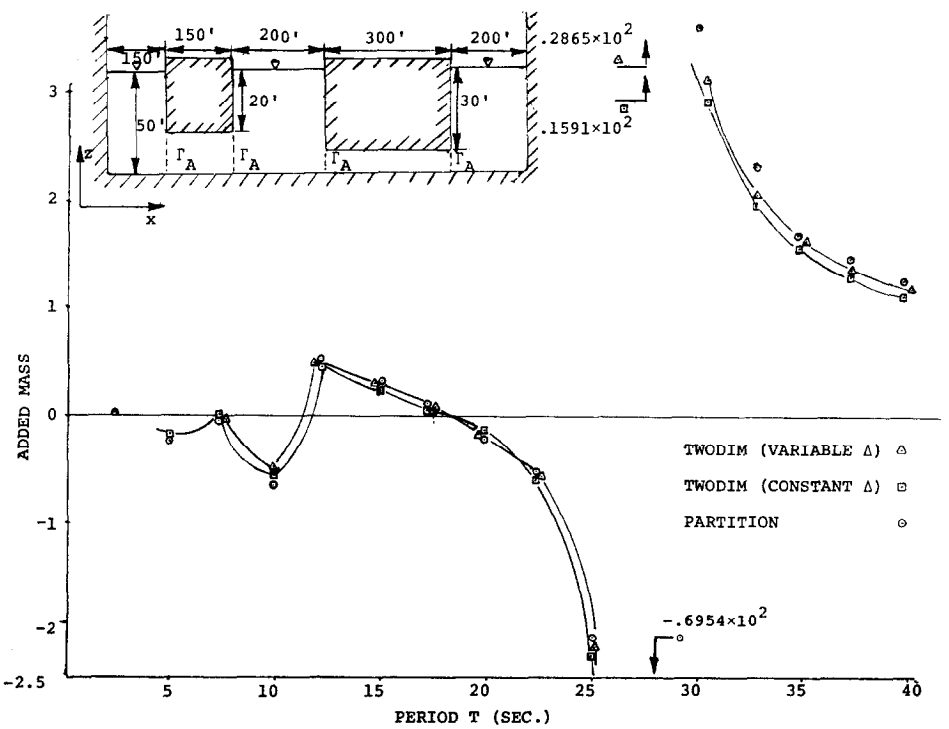
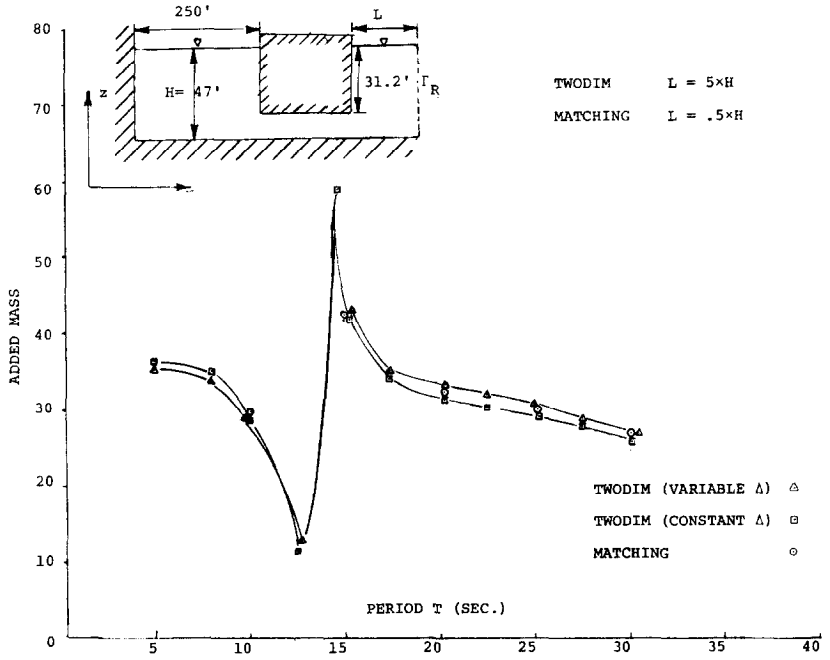


FIG. 4. Sway added mass coefficients for left F. P. with two walls.



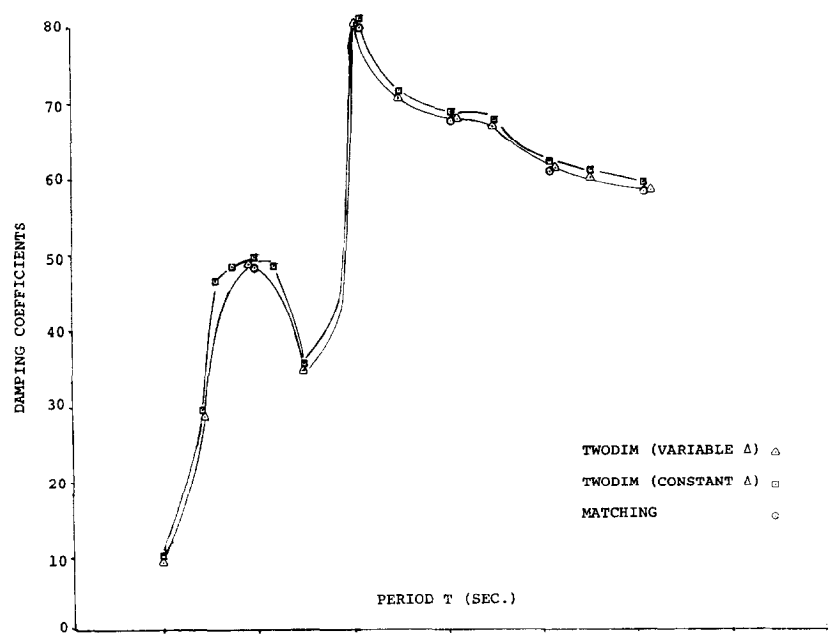


FIG. 5A. Heave added mass factors for one F. P. with one wall. FIG. (B). Heave damping coefficient for one F. P. with one wall.

which are proportional to the added mass coefficient and the damping coefficient which determine the motion of a freely floating body (see [8]). Here Γ_P is the wetted part of the floating platform and V_n is its outward unit normal.

The cases described in Figs. 4 and 5 are those of rectangular platforms. Because of the $3\pi/2$ angles at the bottom of the platforms, the first derivatives of the potential behave like $R^{-1/2}$ in the nieghborhood of these corners, where R is the distance from the corner. None of the programs described here provide any sort of a special expansion for the $3\pi/2$ corners. Figures 4 and 5 show two results of TWODIM: One with a uniform distribution of points and the other with a nonuniform distribution of points such that $\Delta_{j+1/2} = |\mathbf{x}_{j+1} - \mathbf{x}_j|$ is relatively small near corners; the latter gives fairly accurate results even without a special corner treatment. Observe that the results of the partition technique, which was taken with a uniform distribution of points, are more accurate than the corresponding results of TWODIM. This is probably due to the particular choice of the partition lines shown in Fig. 4, which separate the domain into rectangular subdomains with $\pi/2$ angles.

Because of the small variation in the z-direction the number of points along any vertical line is taken to be 7 independent of the frequency of oscillation. The number of points along a horizontal boundary with a length D is taken to be $\max(7, 12\, D/L)$ where L is the asymptotic wavelength $2\pi/k_0$; thus the number of points is approximately proportional to the frequency of oscillation. The gain in using the partition technique versus treating the whole boundary as a single closed contour increases with the complexity of the problem; i.e., it increases with the frequency and a more detailed geometry. In the case shown in Fig. 4, the partition technique was 4-5 times faster than TWODIM in the range of the high frequencies, while in the low frequencies computing time was about the same. The gain in memory is even more significant. For the kind of problems discussed here the partition technique is practically unlimited in the size of problems it can compute in core. TWODIM, on the other hand, could not compute accurately higher frequencies than those shown in Figs. 4 and 5.

The partition and matching techniques presented in this paper provide a very helpful device to tackle large scale problems via integral equation methods. The techniques are generalizable to three dimensional calculations and we feel that using them could make three dimensional problems with complex geometry economically feasible.

Similar techniques can be developed for the reduced wave equation (see [6]).

APPENDIX

In this appendix we describe a method to compute the coefficients of the discrete approximation to (4.9) which is the exterior problem. We use a cubic spline approximation for the potential and its x-derivative which in addition to the smoothness requirements also satisfies the boundary conditions

$$\frac{\partial \phi}{\partial z} = \lambda \phi \qquad \text{for} \quad z = H, \tag{A.1}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{for} \quad z = 0. \tag{A.2}$$

This approximation is given by

$$\frac{\partial \phi}{\partial x}(z) = \sum_{j=1}^{N_R} \frac{\partial \phi}{\partial x}(x_R, z_j)\, S_j(z) + \lambda \frac{\partial \phi}{\partial x}(x_R, H)\left[P(z) - \sum_{j=1}^{N_R} P(z_j)\, S_j(z)\right] \tag{A.3a}$$

where

$$P(z) = z^2(z - H)/H^2. \tag{A.3b}$$

Using the series representation (4.2) we rewrite Eq. (4.10) as

$$\pi\phi(x_R, z_j) = -\lim_{\epsilon \downarrow 0} \int_0^H \gamma^R((x_R + \epsilon, z_j), (x_R, \zeta)) \frac{\partial \phi}{\partial n}(x_R, \zeta)\, d\zeta \tag{A.4}$$

$$= 2\pi i \lim_{\epsilon \downarrow 0} \sum_{m=0}^{\infty} \frac{K_m}{k_m} \cosh k_m z_j \cdot \exp(-|k_m| \epsilon) \int_0^H \frac{\partial \phi}{\partial n}(x_R, \zeta) \cosh k_m \zeta\, d\zeta.$$

We proceed by first substituting the cubic spline approximation (A.3) for $\partial\phi/\partial n$ in the integral in (A.4) and then successively integrating by parts three times. Because both the Green's function and the approximation to the normal derivative (A.3) satisfy the boundary conditions at $z = 0$ and $z = H$, the lower powers of k_m cancel out and we obtain a strongly convergent series in (A.4) even for $\epsilon = 0$. Setting $\epsilon = 0$ and using relation (1.10) we can rewrite (A.4) as

$$\pi\phi_j = -\sum_{l=1}^{N_R} \beta_{j,l} \left(\frac{\partial \phi}{\partial n}\right)_l \tag{A.5a}$$

where

$$\hat{\beta}_{j,l} = -2\pi i \sum_{m=1}^{\infty} \frac{K_m}{k_m^5} \cosh k_m z_j \times \left[S_l'''(0) - \sum_{p=1}^{N_R - 1} S_l''' \left(\frac{\zeta_p + \zeta_{p+1}}{2}\right)(\cosh k_m \zeta_{p+1} - \cosh k_m \zeta_p)\right], \quad 1 \leqslant l \leqslant N_R, \tag{A.5b}$$

$$\beta_{j,l} = \hat{\beta}_{j,l} - \lambda \cdot \delta(x_l, H) \times \left\{\frac{4\pi i}{H} \sum_{m=1}^{\infty} \frac{K_m}{k_m^5} \cosh k_m z_j [\cosh k_m H(2 - 3/H) + 3/H] + \sum_{p \neq l} P(\zeta_p)\, \hat{\beta}_{j,p}\right\}, \tag{A.5c}$$

and

$$\delta(x_l, H) = \begin{cases} 1, & x_l = H, \\ 0, & \text{otherwise.} \end{cases} \tag{A.5d}$$

Since k_m for $m \geqslant 1$ is purely imaginary, $|\cosh k_m z| \leqslant 1$ and therefore the power series in (A.5) are rapidly convergent ($k_m \sim m\pi$). In our computations it was sufficient to take 10 terms.

ACKNOWLEDGMENTS

We thank Professors Peter Lax and Olof Widlund for critically reading this manuscript.

Results presented in this paper were obtained at the Courant Institute of Mathematical Sciences, New York University, under Contract E(11-1)-3077 with the Energy Research and Development Administration.

REFERENCES

1. K. J. BAI, "A Variational Method in Potential Flows with a Free Surface," Ph.D. Dissertation, University of California, Berkeley, September 1972.
2. A. HARTEN, An efficient differentio-integral equation technique for time-dependent potential flows with a free surface *in* "Proceedings of the First International Conference on Numerical Ship Hydrodynamics, Bethesda, Maryland, October 1975."
3. A. HARTEN AND Y. K. CHUNG, "Green's Function Techniques for the Solution of Time-dependent Potential Flows with a Free Surface in a Bounded Domain," ERDA Report COO-3077-100, Courant Inst. of Math. Sci., NYU, December 1975.
4. R. T. HO AND A. HARTEN, On Green's function techniques for solution of floating body problems, *in* "Proceedings of Speciality Conference on Civil Engineering in Oceans, III, University of Delaware, June 1975."
5. F. JOHN, *Comm. Pure Appl. Math.* **3** (1950), 45.
6. J. J. LEE AND F. RAICHLEN, *ASCE J. Waterways, Harbors Coastal Eng. Div.* **98** (1972), 311.
7. J. J. STOKER, "Water Waves," Interscience, New York, 1957.
8. J. V. WEHAUSEN, *Ann. Rev. Fluid Mech.* **3** (1971), 237.
9. R. W. YEUNG, "A Singularity-Distribution Method for Free-Surface Flow Problems with an Oscillating Body," Ph.D. Dissertation, University of California, Berkeley, September 1973.
10. 10. R. W. YEUNG, A hybrid integral-equation method for time-harmonic free-surface flows, *in* "Proceedings of the First International Conference on Numerical Ship Hydrodynamics, Bethesda, Maryland, October 1975."
11. COURANT AND HILBERT, "Methods of Mathematical Physics," Vol. II, Wiley, New York, 1962.